

A Non-conservative, Non-local Approximation of the Burgers Equation

(based on joint work with Shyam Sundar Ghoshal & Emil Wiedemann)

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Introduction

- Consider the following Cauchy problem for a conservation law in quasilinear form:

$$u_t + f'(u)u_x = 0; u(x, 0) = u_0(x).$$

- When $f(u) = u^2/2$, this is the **Burgers equation**; the characteristics solve the ODE

$$\dot{y}(t) = u(y(t), t); y(0) = x_0,$$

which reduces to $y(t) = x_0 + tu_0(x)$; classical solutions generally break down in finite time.

- One way to avoid such blow-up is to introduce a damping term, motivating the **viscous Burgers equation**, a second order PDE:

$$u_t + uu_x = \epsilon \Delta u.$$

The Non-local Approximation

- We can also use the method of characteristics to obtain an **inviscid, first-order approximation**.
- Let η_ϵ be a family of approximate identities and consider the equation

$$u_t + (\eta_\epsilon * u)u_x = 0; u(x, 0) = u_0(x). \quad (1)$$

- A solution of (1) can be viewed as a fixed point of an operator T defined on $L^\infty(\mathbb{R} \times [0, T])$, with $Tg = u$ defined through the linear transport equation

$$u_t + (\eta_\epsilon * g)u_x; u(x, 0) = u_0(x). \quad (2)$$

- For any $g \in L^\infty(\mathbb{R} \times [0, T])$, the transport field $\eta_\epsilon * g$ is uniformly Lipschitz in x , and thus (2) is well-posed.

Motivation

- Non-local equations in **divergence** form, e.g.

$$u_t + ((\eta_\epsilon * u)u)_x = 0,$$

have been of independent interest (Zumbrun 1999).

- The **convective** non-linearity in Burgers is the same as in incompressible fluid equations.
- e.g. Lagrangian Averaged Navier-Stokes- α (LANS- α) models.
- Shock formation can be studied through such equations.
- Unlike KdV regularisations, i.e.

$$u_t + uu_x = -\epsilon u_{xxx},$$

the limit(s) is/are still **weak solutions** of the Burgers equation.

A Priori Estimates

- Equations of this form were earlier considered by Norgard and Mohseni (2008) and Coron, Keimer, and Pflug (2020).
- The general BV case can be obtained by approximation.
- For this, we require some **a priori estimates** to pass to the limit.
- Since the solutions are transported along characteristics, we have **total variation preservation**.

Theorem (Ghoshal, V., and Wiedemann)

Let u be the solution to the Cauchy problem (1) with $u_0 \in BV(\mathbb{R}) \cap C^1(\mathbb{R}) \cap \text{Lip}(\mathbb{R})$. Then, for all $t \geq 0$, $u(t, \cdot) \in BV(\mathbb{R})$ and

$$\|u(t, \cdot)\|_{TV} = \|u_0\|_{TV}.$$

Multidimensional Equation

- The equation (1) is also well-posed in its general, multidimensional versions with C^2 flux, i.e.

$$\partial_t u^\epsilon + (\eta_\epsilon * F'(u^\epsilon)) \cdot \nabla u^\epsilon = 0. \quad (3)$$

- However, the total variation property is weakened to one of mere **boundedness** rather than contraction.

Theorem (Ghoshal, V., and Wiedemann)

The total variation of a smooth solution of (3) at any time is bounded, and in particular

$$\|u^\epsilon(\cdot, x)\|_{TV} \leq \|DY^\epsilon(\cdot, t)\|_\infty^{d-1} \|u_0\|_{TV},$$

where d is the spatial dimension, and Y is the transport map.

- Note that the bound is ϵ -dependent if $d > 1$, which makes the non-local to local limit challenging.

Existence of Weak Solutions

- For initial data in BV , a **weak solution** to the Cauchy problem (3) in $\Omega_T = \mathbb{R}^d \times [0, T)$ is a function $u \in C([0, T); L^1_{loc}(\mathbb{R}^d))$ such that

$$0 = \iint_{\mathbb{R}^d_+} u \left[\partial_t \varphi + \nabla \cdot (\varphi(\eta_\epsilon * F'(u))) \right] dx dt \\ + \int_{\mathbb{R}} u_0(x) \varphi(0, x) dx$$

for all $\varphi \in C_c^\infty(\Omega_T)$.

- $BV \cap L^\infty$ functions can be approximated in L^1 by smooth, Lipschitz functions without increasing the BV norm.
- Hence, the existence of weak solutions to (3) follows from Helly's compactness theorem, L^1_{loc} -Lipschitz continuity in time, and the dominated convergence theorem.

Uniqueness of Weak Solutions

- Weak solutions to (3) can also be shown to be unique.
- Uniqueness follows from their stability in L^1 .

Theorem (Ghoshal, V., and Wiedemann)

Let $u, v \in L^\infty(\Omega_T) \cap BV(\Omega_T)$ be two weak solutions of (3) with initial data u_0, v_0 respectively such that $(u_0 - v_0) \in L^1(\mathbb{R}^d)$. Then, there exists a constant $C_{\epsilon, d} > 0$ depending on ϵ, d such that

$$\|(u - v)(t, \cdot)\|_1 \leq e^{C_{\epsilon, d} t} \|u_0 - v_0\|_1.$$

Note that if $u_0 - v_0 \notin L^1(\mathbb{R}^d)$, then the right-hand side is infinite.

- No entropy, or any additional condition, is required for uniqueness in the BV class.

The Singular Limit: Lipschitz Solutions

- What happens as $\epsilon \rightarrow 0$?
- In general, we can only expect convergence to the entropy solution as $\epsilon \rightarrow 0$ when the limit itself is smooth.

Theorem (Ghoshal, V., and Wiedemann)

Let $u_0 \in \text{Lip}(\mathbb{R}^d)$, $F : \mathbb{R} \rightarrow \mathbb{R}^d$ with $F' \in \text{Lip}(\mathbb{R})$, and let $\tau > 0$ be the maximal time such that the solution u to the local conservation law with initial data u_0 is uniformly Lipschitz on Ω_T for all $T < \tau$.

Then the solutions to the Cauchy problem (3) u^ϵ converge uniformly to u on Ω_T for all $T < \tau$.

- Note that the sequence u^ϵ is **not** a priori known to be precompact through BV bounds if $d > 1$.

The Singular Limit: Riemann Problems

- By the dominated convergence theorem, we can prove that the limit holds in L^1_{loc} up to the time of catastrophe τ .
- The convergence is sharp for general scalar conservation laws.
- Consider the **Riemann problem** in one spatial dimension:

$$u_0(x) = \begin{cases} u_L & \text{if } x \leq 0, \\ u_R & \text{if } x > 0. \end{cases}$$

- The solution is independent of ϵ , given by $u^\epsilon(x, t) = u_0(x - \sigma t)$, where

$$\sigma = \int_{-\infty}^{\infty} \eta_\epsilon(0 - y) f'(u_0(y)) dy = \eta_\epsilon * u_0(0).$$

- For **symmetric** mollifiers, σ is the Rankine-Hugoniot speed for arbitrary u_L, u_R if and only if the flux is quadratic.

The Singular Limit: Weak Solutions

- Analysis of the Riemann problem suggests the following result:

Theorem (Coron, Keimer, and Pflug)

If η_ϵ is symmetric, every convergent subsequence of solutions to the non-local Cauchy problem (1) with BV initial data converges to a *weak* solution of the Burgers equation with the same initial data as $\epsilon \rightarrow 0$.

- Although it is expected that a unique limit exists, even if it is not the entropy solution, this is presently an open problem.
- However, if the initial data is **quasiconcave** and satisfies a positive Lipschitz inequality, i.e. Oleinik's **condition E**

$$u_0(x+h) - u_0(x) \leq Ch$$

for all $h > 0$, then Coron et al show that the (unique) limit is the entropy solution.

A Generic Singular Limit

- Consider initial data $u_0 \in BV(\mathbb{R})$ such that, for $a_1, a_2 \in \mathbb{R}, a_1 < a_2$:

$$u_0(x) = \begin{cases} v_0(x), & \text{if } x < a_1, \\ v_1(x) & \text{if } x \in [a_1, a_2), \\ v_2(x) & \text{if } x \geq a_2, \end{cases}$$

where $v_i(x)$ are uniformly Lipschitz increasing functions with constant C , and the jump discontinuities satisfy

$$u_0(a_1-) > u_0(a_1+), u_0(a_2-) > u_0(a_2+).$$

- It is sufficient to establish **condition E** in order to conclude that the limit is the entropy solution.
- By uniqueness of the limit, the entire sequence converges.

A Generic Singular Limit

- As long as the (limits of) curves from a_1, a_2 do not meet, condition E holds for any sub-sequential limit.
- This can be generalised to the general case of **piecewise Lipschitz increasing functions**.

Theorem (Ghoshal, V., and Wiedemann)

Let $u_0(x) \in BV(\mathbb{R})$ be such that

$$u_0(x) = \begin{cases} v_0(x), & \text{if } x < a_1, \\ v_i(x), & \text{if } x \in [a_i, a_{i+1}), 1 \leq i \leq n-1, \\ v_n(x) & \text{if } x \geq a_n. \end{cases}$$

Then, the non-local solutions u^ϵ of the Cauchy problem (1) with initial data u_0 converge to the entropy solution u of Burgers equation for small enough time.

Isentropic Euler

- We conclude with a simple extension of our results to the **isentropic Euler** equations with pressure law $p(\rho) = \rho^3/3$:

$$\partial_t \rho + \partial_x(\rho v) = 0,$$

$$\partial_t(\rho v) + \partial_x \left(\rho v^2 + \frac{\rho^3}{3} \right) = 0.$$

- The eigenvalues of this system are given by $v \pm \rho$, and the Riemann invariants are given by $\rho \pm v$, and hence for classical solutions we have that

$$(\rho + v) + (\rho + v)\partial_x(\rho + v) = 0,$$

$$(\rho - v) - (\rho - v)\partial_x(\rho - v) = 0.$$

- Thus, we can apply our regularisation schema to this equation, at least in the smooth regime.

References

- *A Non-conservative, Non-local Approximation of the Burgers Equation*; Shyam Sundar Ghoshal, Parasuram Venkatesh, and Emil Wiedemann.
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Thank you!