L^2 Stability of Simple Shocks for Spatially Heterogeneous Conservation Laws

Parasuram Venkatesh

TIFR-CAM

venkatesh2020@tifrbng.res.in

Joint work with Shyam Sundar Ghoshal

1 Introduction

Consider the spatially heterogeneous scalar conservation law

$$\begin{aligned} u_t + f(x,u)_x &= 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{1}$$

Laws of this form arise as simple models of physical phenomena such as traffic flow on a road with smoothly varying maximal velocity. The heterogeneous (x-dependent) case is quite distinct in certain qualitative aspects from the homogeneous (x-independent) one. The unique entropy weak solutions of (1) satisfy

$$\eta(u)_t + Q(x,u)_x + \eta'(u) f_x(x,u) - Q_x(x,u) \leq 0, \tag{2}$$

in the sense of distributions for all pairs of functions η, Q such that the entropy η is a convex function and the entropy flux $Q(x,\cdot)$ is an antiderivative of $\eta'(\cdot) f_u(x,\cdot)$ for each fixed x.

2 Assumptions

For the emergence of a single shock in finite time, we make the following structural assumptions on the C^2 flux f:

1. Stationarity: for some $u_\pm \in \mathbb{R}$ with $u_+ < u_-$ and all $x \in \mathbb{R}$:

$$f_x(x,u_+) = f_x(x,u_-) = 0. \tag{S}$$

2. Uniform Convexity: the flux is uniformly convex in u, i.e. for some $\alpha > 0$:

$$f_{uu} \geq \alpha. \tag{UC}$$

3. Nagumo Growth: f uniformly grows super-linearly in u, i.e. for some symmetric η such that $y^{-1}\eta(y) \to \infty$ as $y \to \infty$, we have that for all x, u:

$$f(x,u) \geq \eta(u). \tag{NG}$$

4. Finite Speed of Propagation: characteristics of the equation propagate with finite speed, i.e. there exists a continuous function θ such that, for all $x \in \mathbb{R}$:

$$|f_u(x,u)| \leq \theta(u). \tag{FSP}$$

In addition to these assumptions, we also require one more condition on the flux to induce the L^2 contraction, namely positive heterogeneity:

$$f_{xu} \geq 0. \tag{P}$$

3 Results

Let $\Phi(x)$ denote the Riemann initial data

$$\phi(x) = \begin{cases} u_- & \text{if } x < 0, \\ u_+ & \text{if } x \geq 0. \end{cases} \tag{3}$$

Under assumptions (S)-(FSP), the entropy solution of the Riemann problem (3) is a simple travelling shock with speed σ satisfying the Rankine-Hugoniot condition

$$\sigma = \frac{f(u_-) - f(u_+)}{u_- - u_+}. \tag{4}$$

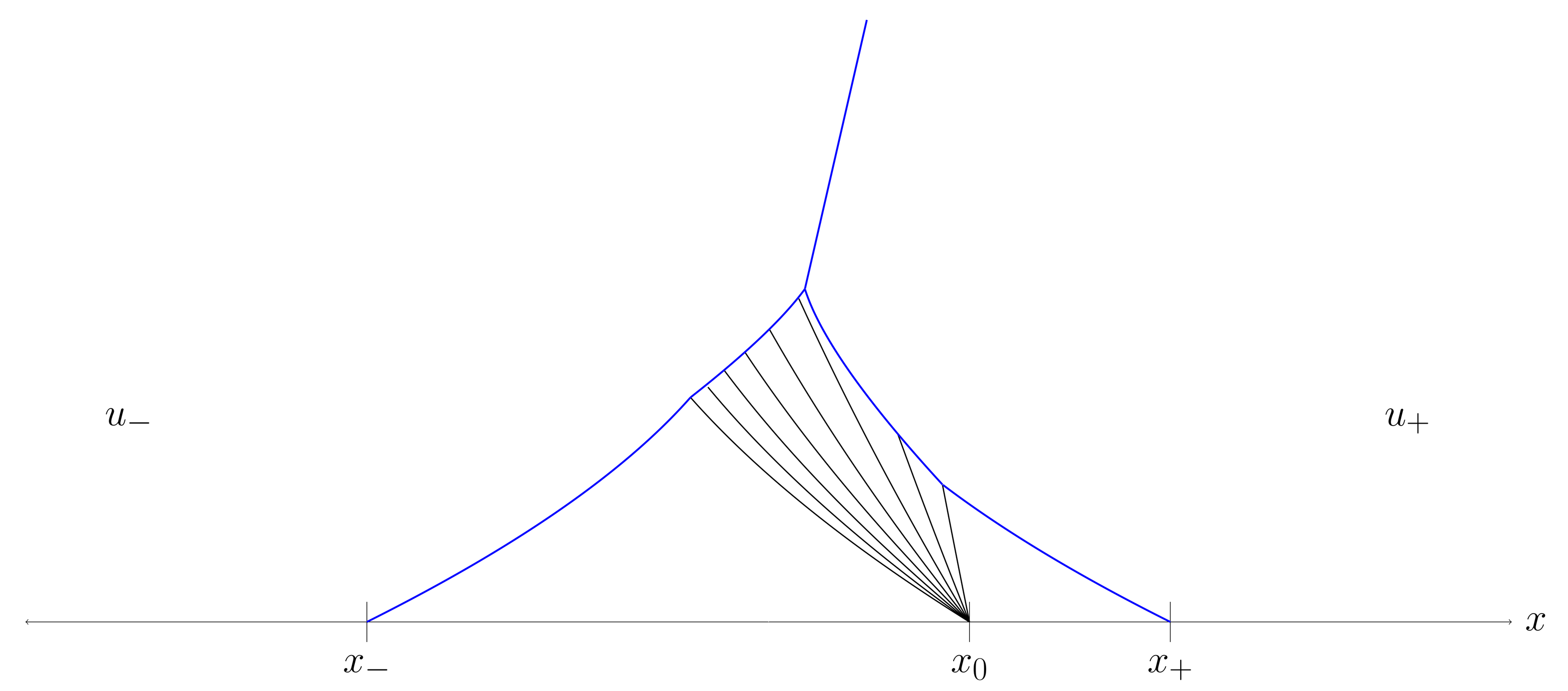
3.1 Emergence of a simple shock in finite time

Under the assumptions S-FSP on the flux, we have the following theorem.

Theorem 3.1. *If $u_0 \in L^\infty(\mathbb{R})$ is such that $u_0 - \Phi$ is compactly supported, then a simple shock emerges in finite time. That is, for some $X, T \in \mathbb{R} \times (0,\infty)$ and all $t > T$:*

$$u(x,t) = \Phi\left(x - (X + \sigma(t-T))\right),$$

where u is the entropy solution of (1) with initial value u_0 and σ is the Rankine-Hugoniot speed (4).



For spatially homogeneous fluxes, this was proved by Dafermos and Shearer [1].

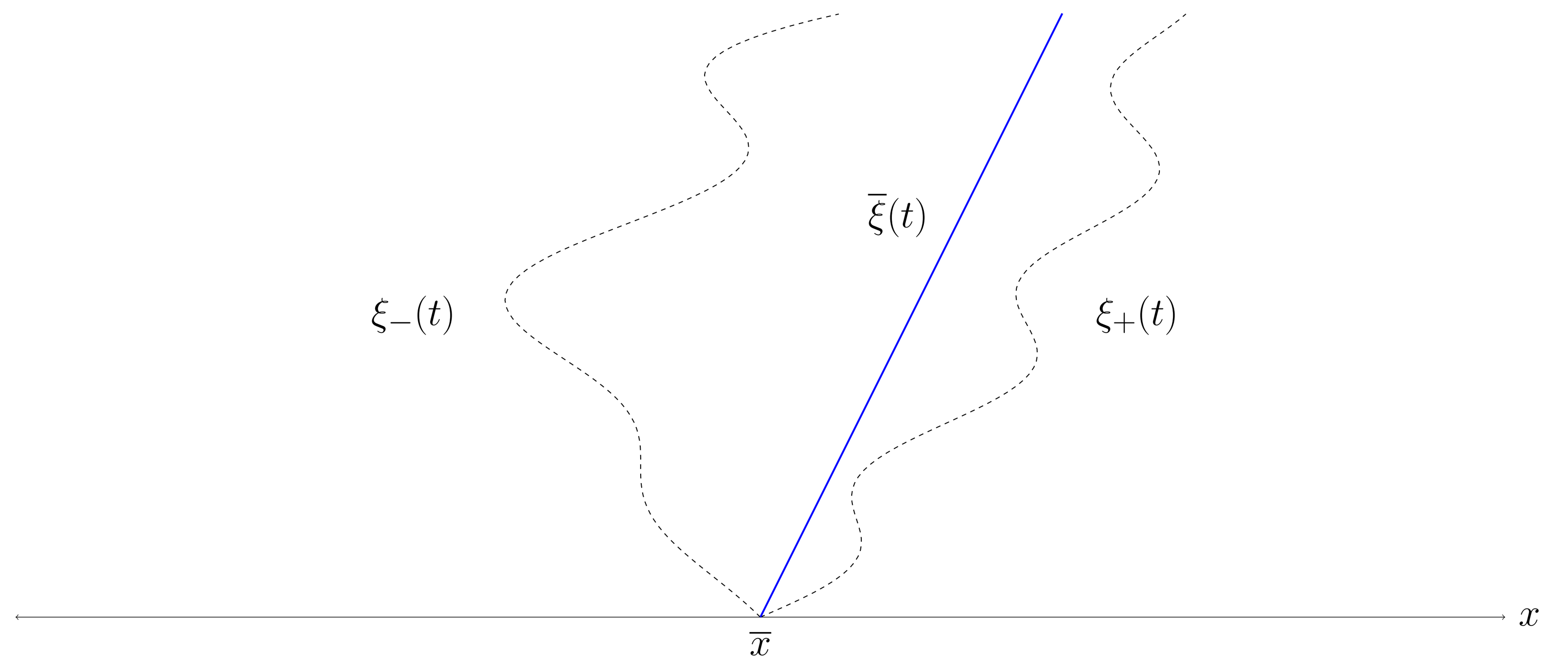
3.2 Stability of the simple shock

Suppose f satisfies P in addition to S-FSP. Then we have the following theorem.

Theorem 3.2. *If $u_0 \in L^\infty$ is such that $\|u_0 - \Phi\|_{L^2} < \infty$, then given any $\overline{x} \in \mathbb{R}$, there exists a Lipschitz curve $\overline{\xi}$ such that $\overline{\xi}(0) = \overline{x}$, and for all $t > 0$:*

$$\left\| u(\cdot,t) - \Phi(\cdot - \overline{\xi}(t)) \right\|_{L^2} \leq \|u_0 - \Phi(\cdot - \overline{x})\|_{L^2},$$

where u is the entropy solution of (1) with initial value u_0.



For spatially homogeneous fluxes, this was proved by Nicholas Leger [3].

4 Counterexamples

The assumptions undergirding Theorems 3.1 and 3.2, in particular S and P, are sharp; we demonstrate this through the following counterexamples.

4.1 Non-stationary solution

Let $u_0 \equiv 1/2$ be constant initial data, for the Cauchy problem (1) with flux

$$f(x,u) = \left(1 + \exp\left(-x^2\right)\right)(u^2 - u), \tag{5}$$

whose characteristic equations are

$$\begin{aligned} \dot{y}(t) &= \left(1 + \exp\left(-y(t)^2\right)\right)(2z(t) - 1), \\ \dot{z}(t) &= 2y(t)\exp\left(-y(t)^2\right)(z(t)^2 - z(t)). \end{aligned} \tag{6}$$

A Lipschitz solution exists at least locally in time. However, for $\epsilon > 0$ small enough, we must have that $z(\epsilon) < 1/2$. Now

$$\dot{y}(\epsilon) = \left(1 + \exp\left(-y(\epsilon)^2\right)\right)(2z(\epsilon) - 1) < 0,$$

thus z remains negative along y, so $\dot{y}(t) \leq \dot{y}(\epsilon) < 0$ for $t \geq \epsilon$, which holds as long as the entropy solution remains shock-free. On the other hand, if $z(0) = 0$, then $\dot{y} \equiv \dot{z} \equiv 0$ from (6). Hence, shocks form in finite time. This example can easily be augmented to demonstrate solutions exhibiting infinitely many shocks from constant initial data by FSP.

4.2 Negative heterogeneity

The non-divergence term in the entropy equation (2) can be written for general C^2 fluxes with a stationary point u_- as

$$\begin{aligned} Q_x(x,u) - \eta'(u) f_x(x,u) &= \int_{u_-}^{u} (y - u_-) f_{xu}(x,y)\,dy - (u - u_-) f_x(x,u) \\ &= -\int_{u_-}^{u} f_x(x,y)\,dy \\ &= -(u - u_-) f_{xu}(x, u_- + \lambda(u - u_-)). \end{aligned} \tag{7}$$

Hence, if $f_{xu}(x_0, u_-) < 0$ for any $x_0 \in \mathbb{R}$, there exists smooth initial data u_0 such that $u_0(x) - u_-$ is compactly supported and non-negative, and for $t > 0$ small enough,

$$\|u(\cdot,t) - u_-\|_{L^2} > \|u_0 - u_-\|_{L^2}. \tag{8}$$

Hence, by (FSP), the simple shock solution of (1) with initial data of the Riemann form is not contractive with respect to L^2 perturbations, even with shifts.

References

[1] Constantine Dafermos and Michael Shearer. Finite time emergence of a shock wave for scalar conservation laws. *Journal of Hyperbolic Differential Equations*, 2010.

[2] Shyam Sundar Ghoshal and Parasuram Venkatesh. L^2 stability of simple shocks for spatially heterogeneous conservation laws, 2025.

[3] Nicholas Leger. L^2 stability estimates for shock solutions of scalar conservation laws using the relative entropy method. *Archive for Rational Mechanics and Analysis*, 2011.

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